Dark Neutrino kinematics

Marco Roda and Iker de Icaza

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Introduction

The differential cross section for COH Dark neutrino interaction provided by Pedro, is differential in the energy of the dark neutrino. The concept is very similar to the CEvNS interaction but since there are a lot of object with masses, the implementation is no trivial and some math formulas need to be specified. This document is a collection of all these assumptions to be referenced by the code.

1 Variables

- $k^\mu = (E_\nu, \boldsymbol{p}_\nu) = (E_\nu, 0, 0, E_\nu)$ incoming neutrino momentum
- $k'^\mu = (E_N, \boldsymbol{p}_N) = (E_N, p_N \sin\theta_N, 0, p_N \cos\theta_N)$ outgoing dark neutrino
- $P^\mu = (M_T, \boldsymbol{0}) = (M_T, 0, 0, 0)$ target momentum
- $P'^\mu = (E_T, \boldsymbol{p}_T) = (E_T, p_T \sin\theta_T, 0, p_T \cos\theta_T)$ recoil nucleus

All the particles are on-shell and so it is valid:

$$\begin{aligned} K^2 &= 0 && (1) \\ K'^2 &= M_N^2 && (2) \\ P^2 = P'^2 &= M_T^2 && (3) \\ k^\mu + P^\mu &= k'^\mu + P'^\mu && (4) \end{aligned}$$

where the last is the momentum conservation between initial and final state.

2 Angle - Energy relations

The cross section is differential in E_N. The final state only has 2 degrees of freedom E_N and ϕ, but this is not enough to build the event as we need $\cos\theta_N$ as well. This is not an independent variable as it depends on E_N. Here we want to evaluate this relation.

Starting from equation(4):

$$\begin{aligned} P'^{\mu} &= k^{\mu} + P^{\mu} - k'^{\mu} && (5) \\ P'^{2} &= k^{2} + P^{2} + k'^{2} + 2k^{\mu}P_{\mu} - 2P^{\mu}k'_{\mu} - 2k^{\mu}k'_{\mu} && (6) \\ M_T^2 &= 0 + M_T^2 + M_N^2 + 2E_{\nu}M_T - 2E_N M_T - 2k^{\mu}k'_{\mu} && (7) \end{aligned}$$

In the lab reference frame we have

$$k^{\mu}k'_{\mu} = E_{\nu}E_N - E_{\nu}p_N\cos\theta_N = E_{\nu}E_N - E_{\nu}\sqrt{E_N^2 - M_N^2}\cos\theta_N \qquad (8)$$

Going on from equation (3)

$$\begin{aligned} M_N^2 + 2E_{\nu}M_T - 2E_N M_T - 2E_{\nu}E_N + 2E_{\nu}p_N\cos\theta_N &= 0 && (9) \\ M_N^2 + 2E_{\nu}M_T - 2E_N(M_T + E_{\nu}) + 2E_{\nu}\sqrt{E_N^2 - M_N^2}\cos\theta_N &= 0 && (10) \end{aligned}$$

Finally the angle at which the dark neutrino opens up in relation to the incoming neutrino is:

$$\cos\theta_N = \frac{E_N(M_T + E_{\nu}) - E_{\nu}M_T - M_N^2/2}{E_{\nu}\sqrt{E_N^2 - M_N^2}} \qquad (11)$$

and the four-momentum of the recoiling nucleus is:

$$P'^{\mu} = k^{\mu} + P^{\mu} - k'^{\mu} = (E_{\nu} + M_T - E_N,\ \boldsymbol{p}_{\nu} - \boldsymbol{p}_N) \qquad (12)$$

3 Validity region

Not every value of E_N is allowed. Clearly $E_N > M_N$ but the kinematic requires more.

3.1 Energy Threshold

Since the outgoing dark neutrino is emitted on-shell with a mass M_N, this gives an energy threshold.

$$\begin{aligned} s|_{Th} &= \left(k'^{*\ \mu}_{Th} + P'^{*\ \mu}_{Th}\right)^2 && (13) \\ (k^{\mu}_{Th} + P^{\mu})^2 &= (M_N + M_T)^2 && (14) \\ 2E_{\nu}^{(Th)}M_T + M_T^2 &= M_N^2 + M_T^2 + 2M_N M_T && (15) \\ E_{\nu}^{(Th)} &= M_N + \frac{M_N^2}{2M_T} && (16) \end{aligned}$$

3.2 Valid angles

We want $\cos\theta_N(E_N) \in [-1;1]$, which corresponds to

$$-1 < \frac{E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2}{E_\nu\sqrt{E_N^2 - M_N^2}} < 1 \tag{17}$$

Or

$$\begin{cases} E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 > -E_\nu\sqrt{E_N^2 - M_N^2} \\ E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 < E_\nu\sqrt{E_N^2 - M_N^2} \end{cases} \tag{18}$$

Luckily this region can be easily identified and it's analytically simple. Figure 1 shows the details of the region constructions. We just need to find the two points of the interval.

The interval extremes are simply given by the intersection of the hyperbola and the line:

$$\begin{aligned} E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 &= \pm E_\nu\sqrt{E_N^2 - M_N^2} &(19)\\ \left(E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2\right)^2 &= E_N^2E_\nu^2 - E_\nu^2M_N^2 &(20)\\ E_N^2(M_T + E_\nu)^2 + (E_\nu M_T + M_N^2/2)^2 - 2E_N(M_T + E_\nu)(E_\nu M_T + M_N^2/2) &= E_N^2E_\nu^2 - E_\nu^2M_N^2 &(21) \end{aligned}$$

which as expected gives us a second order equation in E_N

$$E_N^2(M_T^2 + 2M_TE_\nu) - 2E_N(M_T + E_\nu)(E_\nu M_T + M_N^2/2) + E_\nu^2(M_T^2 + M_N^2) + E_\nu M_T M_N^2 + M_N^4/4 = 0 \tag{22}$$

whose solutions are the usual

$$E_N^\pm = \frac{B \pm \sqrt{B^2 - AC}}{A} \tag{23}$$

and

$$\begin{aligned} A &= M_T^2 + 2M_TE_\nu &(24)\\ B &= (M_T + E_\nu)(E_\nu M_T + M_N^2/2) &(25)\\ C &= E_\nu^2(M_T^2 + M_N^2) + E_\nu M_T M_N^2 + M_N^4/4 &(26) \end{aligned}$$

3.3 The threshold point S

As can be seen by Figure 1 the threshold is a special case as it gives only a single possible value for E_N as a cross check of our understanding, it is worth calculating it. To find the value of E_N^S we could simply get the numbers in the solution, but that would be annoying. Instead we could simply notice that

$$\begin{aligned} E_N &= \gamma_{CM}M_N &(27)\\ E_T &= \gamma_{CM}M_T &(28)\\ \Rightarrow E^{Final} &= \gamma_{CM}(M_N + M_T) &(29)\\ &= E^{inital} = E_\nu^{th} + M_T &(30) \end{aligned}$$

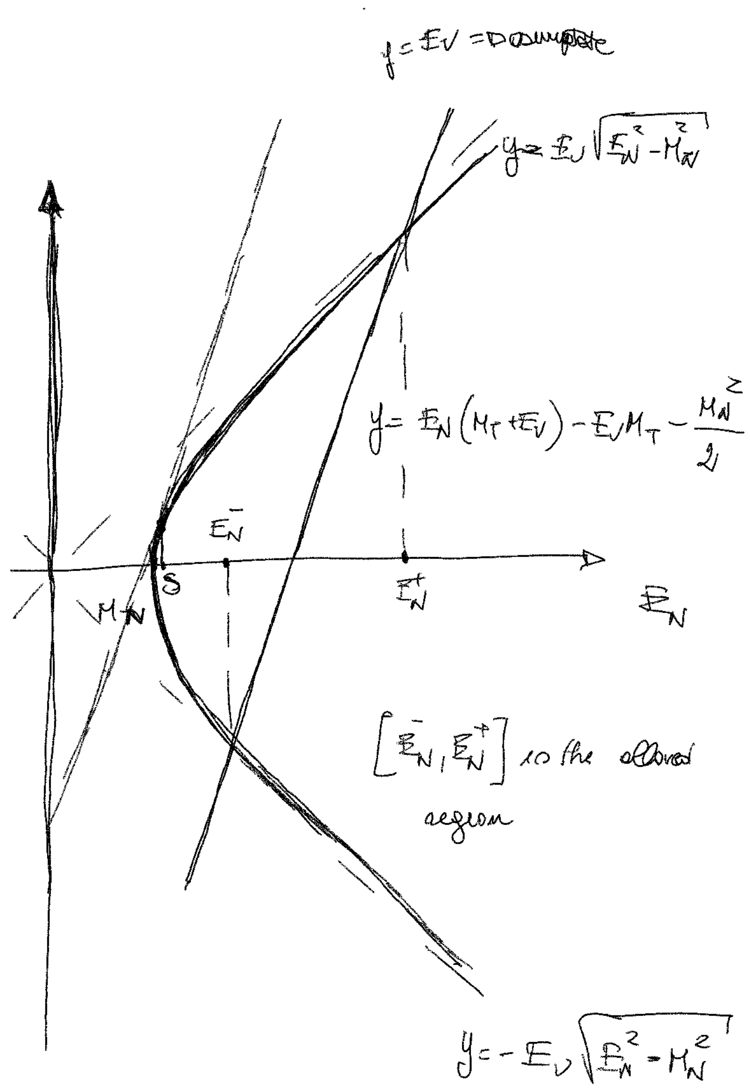

Figure 1: Available E_N region that satisfies equation 18. Even if it is not specified, notice that E_ν in this plot is always bigger than E_N^+

Following equation 16 we then have

$$\gamma_{CM}(M_N + M_T) = M_N + M_T + \frac{M_N^2}{2M_T} \tag{31}$$

and hence

$$\gamma_{CM} = 1 + \frac{M_N^2}{2M_T(M_T + M_N)} \tag{32}$$

$$E_N^S = M_N \left(1 + \frac{M_N^2}{2M_T(M_T + M_N)}\right) \tag{33}$$

3.4 Coherent elastic case

It might be worth reporting here the case for $M_N = 0$ that should clarify the situation. The solution in this case is much simpler, in particular, with respect to Figure 1 the hyperbola turns into its asymptotes:

$$y = \pm E_\nu E_N \tag{34}$$

so the equation of the intersection becomes linear and the solutions are simply

$$E_N^- = \frac{M_T}{2 + \frac{M_T}{E_\nu}} = \frac{E_\nu}{1 + 2\frac{E_\nu}{M_T}} \tag{35}$$

$$E_N^+ = E_\nu \tag{36}$$

The effect of the mass M_N on these limits is restrict the available range for E_N.

4 Differential cross section

According to Pedro, the full differential cross section is

$$\frac{d\sigma}{dE_N} = \frac{e^2 F F^2 \varepsilon^2 \theta^2 g_D^2 M_N Z_T^2 \left(M_N^2(E_N + E_\nu + M_T) - 2M_T(E_N^2 + M_T E_N + E_\nu(E_\nu - M_T))\right)}{8\pi E_\nu^2 M_T \left(M_{Z_D}^2 - 2E_N M_T + 2E_\nu M_T\right)} \tag{37}$$

which has a bunch of terms as a function of M_T: $O(M_T)$, $O(1)$, $O(1/M_T)$. They are:

$$\frac{d\sigma}{dE_N} = \frac{e^2 F F^2 \varepsilon^2 \theta^2 g_D^2 M_N Z_T^2}{8\pi E_\nu^2 \left(M_{Z_D}^2 - 2E_N M_T + 2E_\nu M_T\right)} \left[2M_T\left(E_\nu - E_N\right) + \left(M_N^2 - 2E_N^2 - 2E_\nu^2\right) + \frac{M_N^2}{M_T}\left(E_N + E_\nu\right)\right] \tag{38}$$

So the leading term in M_T is

$$\frac{d\sigma}{dE_N} \simeq \frac{e^2 F F^2 \varepsilon^2 \theta^2 g_D^2 M_N Z_T^2 M_T}{4\pi E_\nu^2 \left(M_{Z_D}^2 - 2E_N M_T + 2E_\nu M_T\right)} \left(E_\nu - E_N\right) \tag{39}$$

The M_T independent term is always negative, the problem is that it usually kills the leading term making the whole cross section negative. We suspect there is a mistake.