Dark Neutrino kinematics

Marco Roda and Iker de Icaza

June 2020

Introduction

The differential cross section for COH Dark neutrino interaction provided by Pedro, is differential in the energy of the dark neutrino. The main reference we have on the subject is [1]. The concept is very similar to the CEvNS interaction but since there are a lot of object with masses, the implementation is no trivial and some math formulas need to be specified. This document is a collection of all these assumptions to be referenced by the code.

1 Variables

- $k^\mu = (E_\nu, \boldsymbol{p}_\nu) = (E_\nu, 0, 0, E_\nu)$ incoming neutrino momentum
- $k'^\mu = (E_N, \boldsymbol{p}_N) = (E_N, p_N \sin\theta_N, 0, p_N \cos\theta_N)$ outgoing dark neutrino
- $P^\mu = (M_T, \boldsymbol{0}) = (M_T, 0, 0, 0)$ target momentum
- $P'^\mu = (E_T, \boldsymbol{p}_T) = (E_T, p_T \sin\theta_T, 0, p_T \cos\theta_T)$ recoil nucleus

All the particles are on-shell and so it is valid:

$$K^2 = 0 \tag{1}$$
$$K'^2 = M_N^2 \tag{2}$$
$$P^2 = P'^2 = M_T^2 \tag{3}$$
$$k^\mu + P^\mu = k'^\mu + P'^\mu \tag{4}$$

where the last is the momentum conservation between initial and final state.

2 Angle - Energy relations

The cross section is differential in E_N. The final state only has 2 degrees of freedom E_N and ϕ, but this is not enough to build the event as we need $\cos\theta_N$ as well. This is not an independent variable as it depends on E_N. Here we want to evaluate this relation.

Starting from equation(4):

$$
\begin{aligned}
P'^{\mu} &= k^{\mu} + P^{\mu} - k'^{\mu} && (5)\\
P'^{2} &= k^{2} + P^{2} + k'^{2} + 2k^{\mu}P_{\mu} - 2P^{\mu}k'_{\mu} - 2k^{\mu}k'_{\mu} && (6)\\
M_T^2 &= 0 + M_T^2 + M_N^2 + 2E_\nu M_T - 2E_N M_T - 2k^{\mu}k'_{\mu} && (7)
\end{aligned}
$$

In the lab reference frame we have

$$
k^{\mu}k'_{\mu} = E_\nu E_N - E_\nu p_N \cos\theta_N = E_\nu E_N - E_\nu \sqrt{E_N^2 - M_N^2}\cos\theta_N \qquad (8)
$$

Going on from equation (3)

$$
\begin{aligned}
M_N^2 + 2E_\nu M_T - 2E_N M_T - 2E_\nu E_N + 2E_\nu p_N \cos\theta_N &= 0 && (9)\\
M_N^2 + 2E_\nu M_T - 2E_N(M_T + E_\nu) + 2E_\nu\sqrt{E_N^2 - M_N^2}\cos\theta_N &= 0 && (10)
\end{aligned}
$$

Finally the angle at which the dark neutrino opens up in relation to the incoming neutrino is:

$$
\cos\theta_N = \frac{E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2}{E_\nu\sqrt{E_N^2 - M_N^2}} \qquad (11)
$$

and the four-momentum of the recoiling nucleus is:

$$
P'^{\mu} = k^{\mu} + P^{\mu} - k'^{\mu} = (E_\nu + M_T - E_N,\ \boldsymbol{p}_\nu - \boldsymbol{p}_N) \qquad (12)
$$

3 Validity region

Not every value of E_N is allowed. Clearly $E_N > M_N$ but the kinematic requires more.

3.1 Energy Threshold

Since the outgoing dark neutrino is emitted on-shell with a mass M_N, this gives an energy threshold.

$$
\begin{aligned}
s|_{Th} &= \left(k'^{*\ \mu}_{Th} + P'^{*\ \mu}_{Th}\right)^2 && (13)\\
(k^{\mu}_{Th} + P^{\mu})^2 &= (M_N + M_T)^2 && (14)\\
2E_\nu^{(Th)}M_T + M_T^2 &= M_N^2 + M_T^2 + 2M_N M_T && (15)\\
E_\nu^{(Th)} &= M_N + \frac{M_N^2}{2M_T} && (16)
\end{aligned}
$$

3.2 Valid angles

We want $\cos\theta_N(E_N) \in [-1;1]$, which corresponds to

$$-1 < \frac{E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2}{E_\nu\sqrt{E_N^2 - M_N^2}} < 1 \tag{17}$$

Or

$$\begin{cases} E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 > -E_\nu\sqrt{E_N^2 - M_N^2} \\ E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 < E_\nu\sqrt{E_N^2 - M_N^2} \end{cases} \tag{18}$$

Luckily this region can be easily identified and it's analytically simple. Figure 1 shows the details of the region constructions. We just need to find the two points of the interval.

The interval extremes are simply given by the intersection of the hyperbola and the line:

$$\begin{aligned} E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2 &= \pm E_\nu\sqrt{E_N^2 - M_N^2} && (19) \\ \left(E_N(M_T + E_\nu) - E_\nu M_T - M_N^2/2\right)^2 &= E_N^2 E_\nu^2 - E_\nu^2 M_N^2 && (20) \\ E_N^2(M_T + E_\nu)^2 + (E_\nu M_T + M_N^2/2)^2 - 2E_N(M_T + E_\nu)(E_\nu M_T + M_N^2/2) &= E_N^2 E_\nu^2 - E_\nu^2 M_N^2 && (21) \end{aligned}$$

which as expected gives us a second order equation in E_N

$$E_N^2(M_T^2 + 2M_T E_\nu) - 2E_N(M_T + E_\nu)(E_\nu M_T + M_N^2/2) + E_\nu^2(M_T^2 + M_N^2) + E_\nu M_T M_N^2 + M_N^4/4 = 0 \tag{22}$$

whose solutions are the usual

$$E_N^\pm = \frac{B \pm \sqrt{B^2 - AC}}{A} \tag{23}$$

and

$$\begin{aligned} A &= M_T^2 + 2M_T E_\nu && (24) \\ B &= (M_T + E_\nu)(E_\nu M_T + M_N^2/2) && (25) \\ C &= E_\nu^2(M_T^2 + M_N^2) + E_\nu M_T M_N^2 + M_N^4/4 && (26) \end{aligned}$$

3.3 The threshold point S

As can be seen by Figure 1 the threshold is a special case as it gives only a single possible value for E_N as a cross check of our understanding, it is worth calculating it. To find the value of E_N^S we could simply get the numbers in the solution, but that would be annoying. Instead we could simply notice that

$$\begin{aligned} E_N &= \gamma_{CM} M_N && (27) \\ E_T &= \gamma_{CM} M_T && (28) \\ \Rightarrow E^{Final} &= \gamma_{CM}(M_N + M_T) && (29) \\ &= E^{inital} = E_\nu^{th} + M_T && (30) \end{aligned}$$

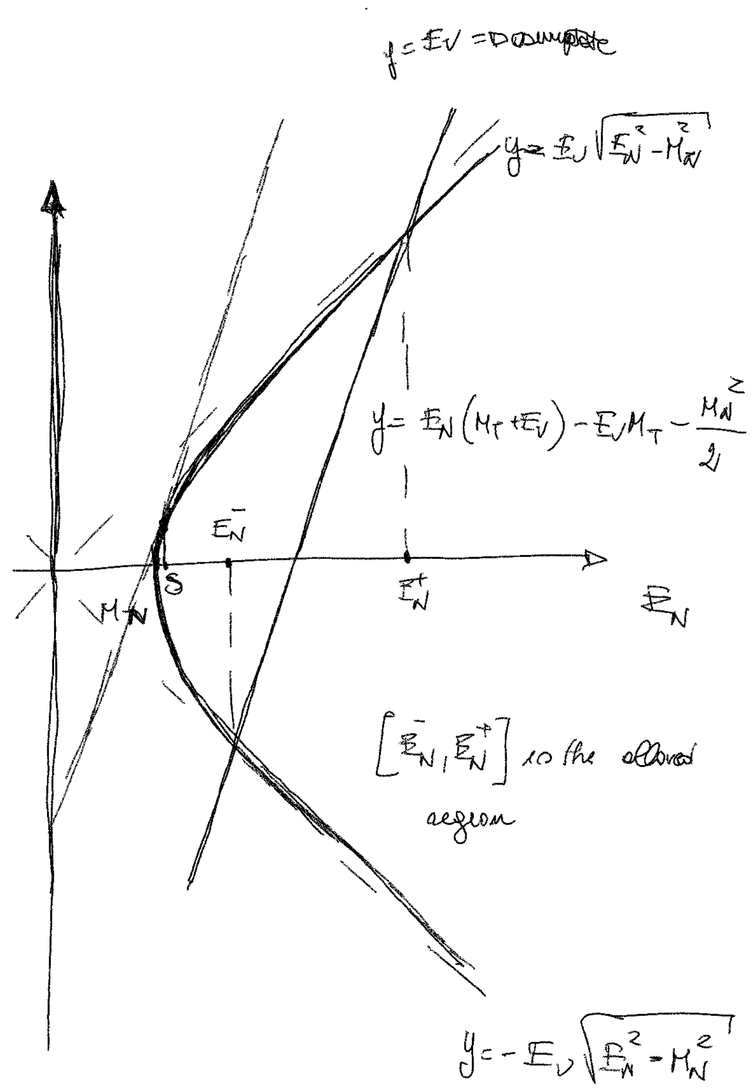

Figure 1: Available E_N region that satisfies equation 18. Even if it is not specified, notice that E_ν in this plot is always bigger than E_N^+

Following equation 16 we then have

$$\gamma_{CM}(M_N + M_T) = M_N + M_T + \frac{M_N^2}{2M_T} \tag{31}$$

and hence

$$\gamma_{CM} = 1 + \frac{M_N^2}{2M_T(M_T + M_N)} \tag{32}$$

$$E_N^S = M_N \left(1 + \frac{M_N^2}{2M_T(M_T + M_N)}\right) \tag{33}$$

3.4 Coherent elastic case

It might be worth reporting here the case for $M_N = 0$ that should clarify the situation. The solution in this case is much simpler, in particular, with respect to Figure 1 the hyperbola turns into its asymptotes:

$$y = \pm E_\nu E_N \tag{34}$$

so the equation of the intersection becomes linear and the solutions are simply

$$E_N^- = \frac{M_T}{2 + \frac{M_T}{E_\nu}} = \frac{E_\nu}{1 + 2\frac{E_\nu}{M_T}} \tag{35}$$

$$E_N^+ = E_\nu \tag{36}$$

The effect of the mass M_N on these limits is restrict the available range for E_N.

4 Differential cross section

4.1 As a function of T_T

The differential cross section as a function of the recoil kinetic energy is:

$$\frac{d\sigma}{dT_T} = \frac{e^2 F F^2 \varepsilon^2 |U_{\mu 4}|^2 |U_{D4}|^2 g_D^2 Z_T^2 \left(M_N^2(T_T - 2E_\nu - M_T) + 2M_T(2E_\nu^2 - 2T_T E_\nu + T_T(T_T - M_T)\right)}{8\pi E_\nu^2 \left(M_{Z_D}^2 + 2T_T M_T\right)^2} \tag{37}$$

$$= \frac{2\pi \alpha_{EM} \alpha_D F F^2 \varepsilon^2 |U_{\mu 4}|^2 |U_{D4}|^2 Z_T^2 \left(M_N^2(T_T - 2E_\nu - M_T) + 2M_T(2E_\nu^2 - 2T_T E_\nu + T_T(T_T - M_T)\right)}{E_\nu^2 \left(M_{Z_D}^2 + 2T_T M_T\right)^2} \tag{38}$$

As in GENIE, what is defined is α_{EM} cross section which is implemented is equation 38.

4.2 As a function of E_N

In order to make this cross section compatible with the generation method based on E_N phase space, we need relation between E_N and T_R. This is:

$$\begin{aligned} T_T &\equiv E_T - M_T \quad (39)\\ &= E_\nu + M_T - E_N - M_T \quad (40)\\ &= E_\nu - E_N \quad (41)\end{aligned}$$

the Jacobian is surprisingly simple as:

$$\begin{aligned} \frac{d\sigma}{dE_N} &\equiv \frac{d\sigma}{dT_R} J \quad (42)\\ &= \frac{d\sigma}{dT_R}\left|\frac{dT_R}{dE_N}\right| \quad (43)\\ &= \frac{d\sigma}{dT_R} \quad (44)\\ \Longrightarrow\ & J = 1 \quad (45)\end{aligned}$$

5 Decay of the dark sector particles

According to [1], the dark neutrino can only decay in a SM neutrino and into a mediator. Both those particles are on-shell. The decay rate is

$$\Gamma_{N\to Z_D+\nu} = \frac{\alpha_D}{2}\left|U_{D4}\right|^2\left(1-\left|U_{D4}\right|^2\right)\frac{m_N^3}{m_{Z_D}^2}\left(1-\frac{m_{Z_D}^2}{m_N^2}\right)\left(1+\frac{m_{Z_D}^2}{m_N^2}-2\frac{m_{Z_D}^4}{m_N^4}\right) \quad (46)$$

then again the mediator will decay into either leptons or SM neutrino according to

$$\begin{aligned} \Gamma_{Z_D\to\nu\bar{\nu}} &= \frac{\alpha_D}{3}\left(1-\left|U_{D4}\right|^2\right)^2 m_{Z_D} \quad (47)\\ \Gamma_{Z_D\to e^+e^-} &\sim \frac{\alpha\epsilon^2}{3} m_{Z_D} \quad (48)\end{aligned}$$

These decays amplitudes are comprehensive and they don't specify the neutrino flavours. Also, equation 48 does not contain a correction factor due to the electron mass. So this section will provide a detail of what is implemented in the code.

5.1 Neutrino flavour

Due to unitarity of the 4×4 mixing matrix, we have

$$\sum_{\alpha=e,\mu,\tau,D}\left|U_{\alpha 4}\right|^2 = 1 \quad (49)$$

so

$$1-\left|U_{D4}\right|^2 = \left|U_{e4}\right|^2+\left|U_{\mu 4}\right|^2+\left|U_{\tau 4}\right|^2 \quad (50)$$

and thus allows us to define the separated decay channels for all the neutrino flavours. Specifically:

$$\Gamma_{N\to Z_D+\nu_\beta} = \frac{\alpha_D}{2}\left|U_{D4}\right|^2\left|U_{\beta4}\right|^2\frac{m_N^3}{m_{Z_D}^2}\left(1-\frac{m_{Z_D}^2}{m_N^2}\right)\left(1+\frac{m_{Z_D}^2}{m_N^2}-2\frac{m_{Z_D}^4}{m_N^4}\right) \tag{51}$$

$$\Gamma_{Z_D\to\nu_\beta\bar{\nu}_\gamma} = \frac{\alpha_D}{3}\left|U_{\beta4}\right|^2\left|U_{\gamma4}\right|^2 m_{Z_D} \tag{52}$$

The paper does not mention it explicitly but the implementation works also for anti-neutrinos. They create an anti-dark-neutrino and the corresponding decay amplitude is the same as equation 51 but in the final state there will be anti-neutrinos. In case it was not obvious at this point, the whole chain of coherent interaction and sequential decay is violating the lepton flavour.

5.2 Phase space correction for electron decay

Equation 48 does not take into account the electron mass. This is a problem because it makes us impossible to generalise this to muons (or tau), so we will have to guess the correction. According to [2, equation 6.35] the 2-body phase space for 2 particles of the same mass m is

$$d\Phi^{(2)} \propto d\Omega\sqrt{1-\frac{4m^2}{M^2}} \tag{53}$$

which is factor we are going to apply to equation 48. And so

$$\Gamma_{Z_D\to e^+e^-} = \frac{\alpha\epsilon^2}{3}m_{Z_D}\sqrt{1-4\frac{m_e^2}{m_{Z_D}^2}} \tag{54}$$

Is the decay rate which is valid for also μ and τ.

5.3 Final remarks

These decay amplitudes are surely not enough to cover all the possible cases in a consistent way.

Since the dark neutrino decay channel is always including dark mediator in the final state, these channels are valid only for $m_N > m_{Z_D}$. The other case would require a dedicated decay channel directly into neutrinos or lepton but we haven't been given the decay amplitude. So the code is checking if the mass hierarchy is respected and if not it calls for a GENIE termination.

A more subtle case is when $m_{Z_D} > m_\pi$. Once the dark boson is heavy enough to produced a pion, there should be channels to allow the pion production, with either neutrinos or electrons in the final state. Again this makes the decay amplitudes we have insufficient to reproduce the correct lifetime of the dark boson, and so the configuration is of the system is considered wrong. For this reason the decay into muons is also incomplete without the hadronic channels.

References

[1] Enrico Bertuzzo et al. "Dark Neutrino Portal to Explain MiniBooNE Excess". In: *Phys. Rev. Lett.* 121 (24 Dec. 2018), p. 241801. DOI: 10.1103/PhysRevLett.121.241801. URL: https://link.aps.org/doi/10.1103/PhysRevLett.121.241801.

[2] Michele Maggiore. *A modern introduction to quantum field theory.* Vol. 12. Oxford university press, 2005.