Revisiting the Rosenbluth EMQE Jacobian

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1 Bug fix for Generator v3.2.0 release

In preparation for the v3.0.6 release of the GENIE Generator product, a change was made to the calculation of a Jacobian used by the Rosenbluth EMQE differential cross section. As described in ref. [1], there was some confusion with respect to the correct Jacobian J to apply when transforming from a cross section differential in Ω, the electron scattering solid angle, to one differential in Q^2:

$$\frac{d\sigma}{dQ^2} = J\,\frac{d\sigma}{d\Omega}\,. \tag{1}$$

Note that the transformation from a two-dimensional to a one-dimensional phase space in eq. (1) is only well-defined because the differential cross section is taken to be uniform in the azimuthal scattering angle ϕ. To make this point explicit, one may use the relation

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\cos\theta\,d\phi} = \frac{1}{2\,\pi}\,\frac{d\sigma}{d\cos\theta} \tag{2}$$

to rewrite eq. (1) as

$$\frac{d\sigma}{dQ^2} = \frac{J}{2\,\pi}\,\frac{d\sigma}{d\cos\theta}\,. \tag{3}$$

By inspection of eq. (3), one may conclude that

$$J = 2\,\pi\,\left|\frac{d\cos\theta}{dQ^2}\right| = 2\,\pi\,\left|\frac{dQ^2}{d\cos\theta}\right|^{-1}\,. \tag{4}$$

In Generator v3.0.4 and earlier, the expression for J used by the `RosenbluthPXSec` class was

$$J = \frac{\pi}{E'^2}\,, \tag{5}$$

where E (E') is the total energy of the initial (final) electron. In version 3.0.6, this was changed to

$$J = \frac{\pi}{E\,E'} \tag{6}$$

based on an argument given in ref. [1]. However, the derivation in ref. [1] makes the assumption that the final electron energy is independent of the scattering cosine $\cos\theta$, which is only true in specific frames of reference, e.g., the center of momentum (CM) frame. Inspection of the source code for the `RosenbluthPXSec` class (`src/Physics/QuasiElastic/XSection/RosenbluthPXSec.cxx`) reveals that the working frame for the cross section calculation is the rest frame of the initial struck nucleon (the "hit nucleon rest" or HNR frame).

Section 2 derives an expression for the Jacobian J in the HNR frame which is found to match that in eq. (5) once the approximations adopted for the Rosenbluth differential cross section are taken into account. The expression in eq. (6), while valid in the CM frame, is incorrect for the HNR frame used by the code.

To fix this error, whose impact is limited solely to simulations of EMQE scattering in Generator v3.0.6, the commit (`fb262552fbec2260fa3b034519fc2f4e7b03cfb2`, see also GitHub pull request #37) introducing the change to the Jacobian has been reverted in preparation for Generator v3.2.0.

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2 Jacobian derivation

Consider quasielastic scattering of an electron on a nucleon with mass M. In the HNR frame, one may write

$$p'^2 = W^2 = (p+q)^2 = M^2 - Q^2 + 2\,M\,(E - E')\,. \tag{7}$$

where p (p') is the 4-momentum of the initial (final) nucleon, W is the mass of the final nucleon, q is the 4-momentum transfer, $Q^2 \equiv -q^2$, and E (E') is the total energy of the initial (final) electron. The use of the HNR frame is assumed throughout the rest of this section.

By implicit differentiation of eq. (7) with respect to the electron scattering cosine $\cos\theta$, one may obtain the relation

$$\frac{dQ^2}{d\cos\theta} = -2\,M\,\frac{dE'}{d\cos\theta}\,. \tag{8}$$

Since the initial nucleon is at rest, the final nucleon 3-momentum is equal to the 3-momentum transfer $\mathbf{q}$. The total energy of the final nucleon E_{N_f} may therefore be written in the form

$$E_{N_f} = \sqrt{|\mathbf{q}|^2 + W^2} = \sqrt{(\mathbf{k}-\mathbf{k}')^2 + W^2} = \sqrt{|\mathbf{k}|^2 - 2\,|\mathbf{k}|\,\sqrt{E'^2 - m_e^2}\,\cos\theta + E'^2 - m_e^2 + W^2} \tag{9}$$

where $\mathbf{k}$ ($\mathbf{k}'$) is the 3-momentum of the initial (final) electron and

$$|\mathbf{k}'| = \sqrt{E'^2 - m_e^2}\,. \tag{10}$$

Energy conservation implies that

$$E' = E + M - E_{N_f}\,. \tag{11}$$

Differentiating both sides of eq. (11) with respect to $\cos\theta$ yields

$$\frac{dE'}{d\cos\theta} = -\frac{dE_{N_f}}{d\cos\theta} = \frac{1}{E_{N_f}}\left(|\mathbf{k}|\,|\mathbf{k}'| - E'\,\frac{dE'}{d\cos\theta} + \frac{|\mathbf{k}|\,E'\,\cos\theta}{|\mathbf{k}'|}\,\frac{dE'}{d\cos\theta}\right)\,, \tag{12}$$

which may be rewritten as

$$\frac{dE'}{d\cos\theta} = \frac{|\mathbf{k}|\,|\mathbf{k}'|^2}{(E' + E_{N_f})\,|\mathbf{k}'| - |\mathbf{k}|\,E'\,\cos\theta} \tag{13}$$

Equation (9) may be rearranged to yield the expression

$$\cos\theta = \frac{W^2 - E_{N_f}^2 + |\mathbf{k}|^2 + |\mathbf{k}'|^2}{2\,|\mathbf{k}|\,|\mathbf{k}'|}\,. \tag{14}$$

Substituting the results from eqs. (13) and (14) into eq. (8) leads to the expression

$$\frac{dQ^2}{d\cos\theta} = -\frac{4\,M\,|\mathbf{k}|\,|\mathbf{k}'|^3}{2\,E_{N_f}\,|\mathbf{k}'|^2 + E'\,(E_{N_f}^2 + |\mathbf{k}'|^2 - |\mathbf{k}|^2 - W^2)} \tag{15}$$

So far, the derivation has been exact. The GENIE implementation of the Rosenbluth EMQE cross section makes two approximations which will now be applied for consistency, leading to a final expression for the Jacobian. First, the electron mass is neglected. Applying the substitutions $|\mathbf{k}| \to E$ and $|\mathbf{k}'| \to E'$ to eq. (15) leads to the result

$$\frac{dQ^2}{d\cos\theta} = -\frac{4\,M\,E\,E'^2}{(E' + E_{N_f})^2 - E^2 - W^2} = -\frac{4\,M\,E\,E'^2}{2\,E\,M + M^2 - W^2}\,, \tag{16}$$

where the substitution $E' + E_{N_f} \to E + M$ (which follows from energy conservation) was used in the second step.

The second approximation is equating the final-state hadronic invariant mass W with the initial nucleon mass:

$$W = M\,. \tag{17}$$

This is exactly true only for a free nucleon target. For quasielastic scattering on bound nucleons, an off-shell value of M is used by GENIE as described in section 3, while W is always taken to be the appropriate on-shell nucleon mass. Neglecting the distinction between the two as in eq. (17) yields the final result

$$\frac{dQ^2}{d\cos\theta} = -2\,E'^2\,, \tag{18}$$

which, via eq. (4), leads immediately to the expression for the Jacobian given in eq. (5).

3 Effective two-body problem

For EMQE scattering on a bound nucleon, 4-momentum conservation can be expressed via the following relation:

$$k + p_i = k' + p' + p_f \tag{19}$$

where k (k') is the 4-momentum of the initial (final) electron, p_i (p_f) is the 4-momentum of the initial (final) nucleus, and p' is the 4-momentum of the ejected nucleon. I neglect final-state interactions because they are not of interest for the present discussion.

In the laboratory frame (the rest frame of the initial nucleus), eq. (19) implies the following energy balance:

$$E + m_i = E' + E_{N_f} + m_f + T_f. \tag{20}$$

Here I've split the total energy of the final nucleus into its mass m_f and its kinetic energy T_f for later convenience. The symbol m_i denotes the mass of the initial nucleus. Rearranging eq. (20) and adding m_{N_i}, the on-shell mass of the initial struck nucleon, to both sides allows one to write

$$E + m_{N_i} = E' + E_{N_f} + E_B + T_f\,. \tag{21}$$

The "binding energy" E_B is defined as the *removal energy*

$$E_B \equiv m_f + m_{N_i} - m_i. \tag{22}$$

Inspection of eq. (21) allows one to express the energy balance as if the EMQE reaction were just elastic scattering of an electron on a solitary nucleon:

$$E + E_{N_i} = E' + E_{N_f} \tag{23}$$

Here, the total energy of the initial nucleon E_{N_i} is off the mass shell and is given in the lab frame by

$$E_{N_i} = m_{N_i} - E_B - T_f. \tag{24}$$

By definition, the final nucleus participates in EMQE only as a spectator. Since the initial nucleus is at rest in the lab frame, the 3-momenta of the final nucleus and the initial struck nucleon must be be equal and opposite ($\mathbf{p}_f = -\mathbf{p}_{N_i}$). It follows from this observation and eqs. (19) and (23) that

$$k + p = k' + p'\,, \tag{25}$$

where $p \equiv (E_{N_i}, \mathbf{p}_{N_i})$ is the 4-momentum of the initial struck nucleon. That is, QE scattering may be treated as a $2 \to 2$ process if an off-shell total energy E_{N_i} is assigned to the initial struck nucleon. This corresponds to an off-shell initial nucleon mass M which may be computed via

$$M = \sqrt{p^2}\,. \tag{26}$$

It should be noted that the lab-frame kinetic energy T_f of the final nucleus may be written in terms of quantities that are fully determined by the initial state:

$$T_f = \sqrt{m_f^2 - |\mathbf{p}_f|^2} - m_f = \sqrt{(m_i + E_B - m_{N_i})^2 - |\mathbf{p}_{N_i}|^2} - m_i - E_B + m_{N_i}. \tag{27}$$

GENIE generates QE events by taking $\mathbf{p}_{N_i}$ and E_B as input from the nuclear model. The initial nucleon energy E_{N_i} is set to be off-shell according to the prescription shown above, and the remaining kinematics are then selected.

References

[1] A. Papadopoulou and M. Roda, "Rosenbluth PXSec fix." GENIE docDB #133.